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LOW RANK MODIFICATIONS OF JACOBI AND JOR ITERATIVE METHODS

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Abstract—The purpose of this paper is to introduce new iterative methods for the solution of linear systems, which improve the convergence rate of either Jacobi or JOR (Jacobi over-relaxation) methods. Such methods are well suitable for parallel implementation.

1. INTRODUCTION

Let us consider the linear system $Ax = b$, where A is a real $n \times n$ matrix with nonzero diagonal part D.

The purpose of this paper is to introduce low rank modifications of Jacobi and Jacobi over-relaxation (JOR in the following) methods, leading to iterative methods with improved convergence rates. We define the matrix associated to the rank k modification of the Jacobi method as

$$P_k = J + UV^{\mathrm{T}}A,$$

where U and V are $n \times k$ matrices, $k \ll n$, and $J = I - D^{-1}A$.

We will suggest a choice of U and V for which the spectral radius of P_k [$r(P_k)$ in the following] is equal to $|m_{k+1}|$, if m_i, $i = 1, \ldots, n$, are the eigenvalues of J, and $|m_1| \geqslant |m_2| \geqslant \ldots \geqslant |m_n|$.

In the case $k = 1$ it is possible to prove that such a choice minimizes the Frobenius norm of the matrix P_1. Moreover, we extend our analysis to the JOR method.

Our method appears to be very attractive both for problems leading to associated Jacobi matrices with few eigenvalues with absolute value $\geqslant 1$, and for problems leading to convergent Jacobi matrices with few eigenvalues whose absolute value is close to the spectral radius.

Our approach allows deriving, in the former case, a convergent iterative scheme and in the latter case, an improved convergence rate, with a computational effort not greater than that implied by relaxation methods.

An example of the former situation is given by the discrete approximation of a class of Sturm–Liouville equations, namely

$$\begin{cases} -\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x) \text{ on } (0, 1), \\[2ex] u(0) + \dfrac{\mathrm{d}u(0)}{\mathrm{d}x} = 0, \quad u(1) = 0. \end{cases} \tag{1}$$

Using a standard difference approximation for expression (1), it turns out that the associated Jacobi has a spectral radius equal to one, attained by two eigenvalues.

On the other hand, in the case of convergent Jacobi matrices with clustered eigenvalues near the spectral radius, our approach implies a nontrivial computational effort to obtain a practical acceleration.

The paper is organized as follows. In the next section we define with more details the low rank modification of a given iterative method; in Section 3 we suggest a choice of matrices U and V, and we give some theoretical results about the convergence of our method. In Section 4 we show the parallel complexity of the algorithm presented in this paper.

2. THE ITERATIVE METHOD

Let $A = (a_{ij})$ be a real $n \times n$ nonsingular matrix. We will use the following notations:

$D = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ is the diagonal part of A; given a vector norm $\|.\|$, $\|A\|$ is the induced matrix norm; $\|A\|_F$ is the Frobenius norm of A, i.e.

$$\|A\|_F = \left(\sum_{ij}^{n} a_{ij}^2\right)^{1/2}.$$

Let us consider the splitting $A = M - N$ of A, leading to an iterative method with associated matrix $P = I - M^{-1}A$. We construct a splitting of the type $A = M_1 - N_1$, where

$$M_1^{-1} = M^{-1} - uv^T,$$

with u and v suitable n-vectors, verifying

$$v^T M u \neq 1.$$

Therefore, the linear system $Ax = b$ can be written as follows:

$$x = (P + uv^T A)x + M^{-1}b - uv^T b, \tag{2}$$

from which, by setting

$$P_1 = P + uv^T A, \tag{3}$$

the iterative method

$$x_{i+1} = P_1 x_i + M^{-1}b - uv^T b \tag{4}$$

can be derived.

The technique described above can be generalized, by considering splittings of the type

$$A = M_k - N_k,$$

where

$$M_k^{-1} = M^{-1} - UV^T,$$

with U and V $n \times k$ matrices, $k \ll n$, such that $V^T M U - I$ is nonsingular. We obtain

$$x_{i+1} = P_k x_i + M^{-1}b - UV^T b, \tag{5}$$

where

$$P_k = (P + UV^T A). \tag{6}$$

3. CHOICE OF THE PARAMETERS AND ANALYSIS OF THE CONVERGENCE

In this section we consider linear systems with symmetric coefficient matrix A, having a positive diagonal part. Moreover, we assume A to have a unit diagonal. Following Refs [1, 2], it is easy to show that this assumption is not restrictive. Indeed the matrix $H = SAS$, where $S^2 = D^{-1}$, is symmetric, its diagonal entries are equal to 1, and it is similar to $D^{-1}A$.

In the rest of this section, we analyze the case $P = J$, where J is the matrix associated to the Jacobi iterative method.

It is easy to prove the two following propositions.

Proposition 3.1

Let J be the Jacobi matrix, and P_1 be the matrix defined in expression (3). Let m_i, $i = 1, \ldots, n$, be the eigenvalues of J, and assume that $|m_1| \geqslant |m_2| \geqslant \cdots \geqslant |m_n|$. If v is chosen as an eigenvector of J associated to the spectral radius, and $u = -JAv/v^T A^2 v$, then $r(P_1) = |m_2|$.

More generally, we can state the following.

Proposition 3.2

Let J be the Jacobi matrix, and P_k be the matrix defined in expression (6). Let m_i, $i = 1, \ldots, n$, be the eigenvalues of J, and assume that $|m_1| \geqslant |m_2| \geqslant \ldots \geqslant |m_n|$. If the columns v_i of V are orthogonal eigenvectors of J associated to $m_1, m_2, \ldots, m_k$, and the columns u_i of U are defined as $-JAv_i/v_i^T A^2 v_i$, $i = 1, \ldots, k$, then $r(P_k) = |m_{k+1}|$.

Remark 3.1

Note that when matrix J has p eigenvalues with absolute value $\geqslant 1$, $0 \leqslant p \leqslant k$, then Proposition 3.2 implies that matrix (6) is convergent.

The following results show that the choices for u and v suggested above yield the minimum of the Frobenius norm of the matrix P_1.

Proposition 3.3

Let B be an $n \times n$ matrix, z and w be n-vectors. Then the minimum of the function $f(z, w) = \| B - zw^T \|_F^2$, when z and w vary in R^n, is $\| B \|_F^2 - r(B^T B)$, and it is attained when w is an eigenvector of $B^T B$ associated to the spectral radius, and $z = Bw/w^T w$.

See the Appendix for the proof.

Let now

$$g(u, v) = \| I - A + uv^T A \|_F^2.$$

We have the following corollary.

Corollary 3.1

Let $J = I - A$ be the Jacobi matrix, associated to A.

Let u and v be n-vectors. Then the minimum of $g(u, v)$, when u and v vary in R^n, is

$$\| J \|_F^2 - r(J^2),$$

and it is attained when v is an eigenvector of J associated to the spectral radius, and

$$u = -JAv/v^T A^2 v.$$

Proof. The proof trivially follows from Proposition 3.3 and from the fact that A is nonsingular, and A and J have the same set of eigenvectors. ■

We take now into account the JOR method, for which we have the following.

Proposition 3.4, Ref. [3]

Let A be an $n \times n$ positive definite matrix, and let l_i, $i = 1, \ldots, n$, be the eigenvalues of A in descending order. Then

$$\min_\omega r(I - \omega A) = (l_1 - l_n)/(l_1 + l_n),$$

and it is attained for $\omega = 2/(l_1 + l_n)$.

Proof. The proof follows from the equality

$$r(I - \omega A) = \max\{|1 - \omega l_1|, 1 - \omega l_n|\}.$$ ■

Corollary 3.2

Let $J(\omega)$, $P_1(\omega)$ and $P_2(\omega)$ be the matrices associated to the optimal JOR, rank 1 and rank 2 modifications of the optimal JOR method, respectively. Then:

(1) $r[P_1(\omega)] = r[J(\omega)]$;

(2) if A has simple extreme eigenvalues, then $r[P_2(\omega)] < r[J(\omega)]$.

Proof. The proof follows from Proposition 3.1, and from the fact that at least two eigenvalues of A attain the spectral radius of $J(\omega)$.

Finally, it is worth comparing the rank 1 modification of the Jacobi method to the optimal JOR. We have the following.

Remark 3.2

Let $\omega = 2/(l_1 + l_n)$. It is easy to prove that $r[J(\omega)] > r(P_1)$, if $|m_2|$ is given by min $\{|1 - l_1|, |1 - l_n|\}$. Otherwise, no general relation between the spectral radii can be stated.

Moreover, it will be clearer later that our method requires the same computational effort as the optimal JOR.

4. PARALLEL IMPLEMENTATION AND CONCLUSIONS

In this section we are concerned with the parallel complexity of the algorithm described in the above sections. Consider first the rank 1 modification of the Jacobi method.

Let $l_1 \geqslant l_2 \geqslant \ldots \geqslant l_n > 0$, and $m_{j_i} = 1 - l_i$, $i = 1, \ldots, n$, where j_i, $i = 1, \ldots, n$, is a permutation of $1, 2, \ldots, n$, such that $|m_1| \geqslant |m_2| \geqslant \cdots \geqslant |m_n|$, be the eigenvalues of A and J, respectively.

It is useful to make the following observation.

Remark 4.1

The maximum eigenvalue of $B = I - 1/dA$, where d is a constant $\geqslant l_1$, is $\eta = 1 - l_n/d$.

A practical choice for d is given by $d = \| A \|$.

Then it is possible to compute successfully l_1 and l_n (and hence m_1) by the power method Ref. [4] applied to the positive definite matrices A and $I - 1/\| A \| \, A$. Indeed, note that the condition of positive definiteness assures the convergence of the power method, by avoiding the presence of distinct eigenvalues with the same absolute value.

One step of the power method can be performed with

$$2\lceil \log n \rceil + 2 \text{ time on } n^2 \text{ processors.}$$

Following Ref. [4], it is possible to show that the number of steps required by the power method to compute l_1 and the corresponding eigenvector is $O[1/\log(l_1/l_r)]$, where r is the least integer for which $l_r < l_1$.

Therefore the iterative method (4) can be performed according to the following stages.

		Steps	Processors
1.	Compute v	$4c(\lceil \log n \rceil + 1)$	$O(n^2)$
2.	Compute P_1	$2(\lceil \log n \rceil + 1)$	$O(n^2)$
3.	Compute $M_1^{-1}b$	$4(\lceil \log n \rceil + 1)$	$O(n^2)$
4.	Compute x_k	$k(\lceil \log n \rceil + 1)$	$O(n^2)$

The overall running time of the algorithm is given by

$$(4c + K + 6)(\lceil \log n \rceil + 1), \text{ on } O(n^2) \text{ processors,}$$

where c and k are the number of steps of the power method and the iterative method, respectively.

When the spectral radius of J is attained by r eigenvaleus, $r > 1$, then it is necessary to perform a rank k modification, with $k \geqslant r$, in order to improve the convergence rate. For this purpose it is possible either to use the power method with deflation, or to use the Lanczos method (Ref. [5]), which requires low parallel computational efforts when applied to compute a few extreme eigenvalues.

Analogous arguments can be stated for the modification of the optimal JOR method.

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APPENDIX

Proof of Proposition 3.3

The function $f(z, w)$ has minimum in R^{2n}. Indeed, not that $f(z, w) = f(lz, 1/l\, w)$, for any nonzero real constant l. From the above arguments, it follows that the condition $\| w \|_2 = 1$ is not a restriction, and we can study the behavior of f on the set $\{(z, w) \in R^{2n}: \| w \|_2 = 1\}$. Moreover for any real positive constant K, there exists a quantity M such that if $z_i > M$, $i = 1, \ldots, n$, then $f(z, w) > K$.

The set $S_M = \{(z, w) \in R^{2n}: \| w \|_2 = 1, z_i < = M, i = 1, \ldots, n\}$ is closed and bounded, $f(z, w)$ is a continuous mapping, and hence there exists the minimum of f, when z and w vary in S_M, for any M.

If we choose a suitable value for K, and consequently for M, from the above arguments it follows that such a minimum is a minimum of f on R^{2n}.

Now we compute the stationary points of f, among which we look for the one minimizing the value of f.

We have

$$\frac{\partial f(z, w)}{\partial z_p} = -2 \sum_{j=1}^{n} (b_{pj} - z_p w_j) w_j, \quad p = 1, \ldots, n,$$

and

$$\frac{\partial f(z, w)}{\partial w_q} = -2 \sum_{i=1}^{n} (b_{iq} - z_i w_q) z_i, \quad q = 1, \ldots, n,$$

from which,

$$\begin{cases} Bw - w^{\mathrm{T}}wz = 0, \\ B^{\mathrm{T}}z - z^{\mathrm{T}}zw = 0, \end{cases}$$

and the thesis readily follows. ■